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VARIATIONAL PRINCIPLE FOR GUN DYNAMICS WITH ADJOINT VARIABLE FORMULATION

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Gun dynamics problems involving a moving shell have several delta functions in the forcing terms of the equations of motion. The use of a variational method in conjunction with finite elements smooths the differentiability of the variables in the expression involving the delta functions. This report suggests that solutions of the gun dynamics problems be obtained numerically by a variation principle where the far end conditions in time are not required
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20. ABSTRACT (CONT'D)

for purposes of computation. In solving mixed boundary and initial value problems of a high order partial differential equation using spline functions, the computation may be simplified considerably if the variable in time can be truncated into arbitrary sections. Each section may have several node points for the spline functions in the time domain. This is true because we found from previous papers that the initial value problem can be solved in one direction using variational principle and cubic Hermite Polynomials, without worrying about the conditions at the far end.

The end conditions of the ajoint system can be adjusted according to the end conditions of the original system so that the bilinear concomitant is identically zero. This satisfies the variational principle. A bilinear form of the original and adjoint variables is employed in determining the coefficients of the variations of the functions and their derivatives. For the spatial variables Hermite Polynomial spline functions will be used. Algorithm and procedure of computation are given.

The variational principle for spatial and temporal problems with boundary and initial conditions are investigated. This variational principle is very general in scope and can be applied to many linear partial differential equations. The Euler-Bernoulli beam equation satisfies these variational principles. This lays the foundation for gun dynamics problems to be studied.

TABLE OF CONTENTS

INTRODUCTION

This report discusses the use of adjoint variable formulation to seek the transient solutions for problems in gun dynamics. The theory from variational principle involving adjoint variables solves a mixed boundary and initial value problem. The partial differential equation governing the motion has a fourth order partial in spatial domain and a second order partial in time domain. It also involves a few step functions and delta functions as follows.[1,2]

$$\rho A\ddot{y} + (EIy'')'' - [P(x,t)y']' + Ty''H(x-x_p) = m[\dot{x}_p{}^2 y'' + 2\dot{x}_p\dot{y}' + \ddot{y}]\delta(x-x_p) - mg\cos\alpha\,\delta(x-x_p) - \rho Ag\cos\alpha \quad (1)$$

The above equation can be simplified into the following form

$$Ly + Q = 0 \quad (2)$$

where

$$Ly = (\alpha y_t)_t - (\lambda y_{xx})_{xx} + (\ell y_x)_x + (\ell_p y_x)_x H(x-x_p) \quad (3)$$

and

$$-Q = m[\dot{x}_p{}^2 y'' + 2\dot{x}_p\dot{y}' + \ddot{y}]\delta(x-x_p) - mg\cos\alpha\,\delta(x-x_p) - \rho Ag\cos\alpha \quad (4)$$

We seek the explicit numerical transient solutions of y, y_t, y_x, y_{xt}, y_{xx}, and y_{xxt} for some given boundary and initial conditions. The term y_{xx} will give the stress wave and the term y_x will show the slope in bending, along

[1]Simkins, Thomas E., "Transverse Response of Gun Tubes to Curvature-Induced Load Functions," presented at the Second US Army Symposium on Gun Dynamics, Watervliet, NY, September 1978.
[2]Wu, Julian, "Gun Dynamic Analysis by the Use of Unconstrained, Adjoint Variational Formulations," presented at the Second US Army Symposium on Gun Dynamics, Watervliet, NY, September 1978.

the axis of the gun tube. The solution is the extension of our previous work on initial and boundary problems.[3,4]

VARIATIONAL PRINCIPLE USING ADJOINT VARIABLE

If the inner product of the variable y, and the adjoint forcing function $\bar{Q}$ are used for variational purposes, the accuracy is much less than the method using the following inner product by adding a term involving the adjoint variable $\bar{y}$ as the Lagrange multiplier (see Appendix).

$$J[\bar{y},y] = \langle \bar{Q},y\rangle + \langle \bar{y},(Q+Ly)\rangle = 0 \tag{5}$$

where the partial differential equation is given in Eq. (2). By taking variation on Eq. (5) we have

$$\delta J = \langle \delta\bar{y},(Ly+Q)\rangle + \langle \delta y,(\bar{L}\bar{y}+\bar{Q})\rangle - \langle \delta y,\bar{L}\bar{y}\rangle + \langle \bar{y},L\delta y\rangle = 0 \tag{6}$$

The above variation vanishes if

$$Ly + Q = 0 \tag{7}$$

$$\bar{L}\bar{y} + \bar{Q} = 0 \tag{8}$$

and

$$\langle \bar{y},L\delta y\rangle - \langle \delta y,\bar{L}\bar{y}\rangle = 0 \tag{9}$$

We know that Eq. (7) is actually the original p.d.e. and Eq. (8) is its adjoint equation. A method should be established so that Eq. (9) holds true for all arbitrary variation δy.

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[3] Shen, C. N. and Wu, J. J., "A New Variational Method for Initial Value Problems Using Piecewise Hermite Polynomial Spline Functions," presented at the 1981 Army Numerical Analysis and Computer Conference, Huntsville, AL, February 1981.

[4] Shen, C. N., "Method of Solution for Variational Principle Using Bicubic Hermite Polynomial," presented at the 27th Conference of Army Mathematicians, West Point, NY, June 1981.

BILINEAR CONCOMITANT

We will find out the conditions for the assumed equality in Eq. (9) to be true. Let us consider the following bilinear concomitant:[5]

$$D = \langle \bar{y}, Ly \rangle - \langle y, \bar{L}\bar{y} \rangle \tag{10}$$

The above expression can be integrated in two different ways and can also be written in terms of boundary conditions and initial conditions. It is assumed that these boundary conditions are assigned in such a manner that the above bilinear concomitant is identically zero for all independent variables, i.e.,

$$D \equiv 0 \tag{11}$$

Then the first variations of D also vanish.

$$\delta D = \delta D(\delta \bar{y}) + \delta D(\delta y) = 0 \tag{12}$$

Since $\delta \bar{y}$ and δy are independent of each other, then

$$\delta D(\delta \bar{y}) = \langle \delta \bar{y}, Ly \rangle - \langle y, \bar{L}\delta \bar{y} \rangle = 0 \tag{13}$$

$$\delta D(\delta y) = \langle \bar{y}, L\delta y \rangle - \langle \delta y, \bar{L}\bar{y} \rangle = 0 \tag{14}$$

Equation (14) is identical to Eq. (9), which is the assumed equality previously. The implication is that if Eq. (11) is true then Eq. (9) or (14) is automatically true.

Since Eq. (10) can be expressed in terms of some integrals involving boundary conditions, Eq. (11) can be true if these boundary conditions are satisfied. The next section will discuss integral of bilinear expression and its boundary conditions.

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[5] Stacey, W. M. Jr., Variational Methods in Nuclear Reactor Physics, Academic Press, 1974.

INTEGRAL OF BILINEAR EXPRESSION

The integral of a bilinear expression for a two dimensional problem having second order partial derivatives in time and fourth order partial derivatives in space can be written as

$$I = \int_{x_o}^{x_b} \int_{t_o}^{t_b} \Omega[y(x,t)\bar{y}(x,t)]dtdx \tag{15}$$

where $\Omega[y,\bar{y}]$ is a given bilinear expression in the form

$$\Omega[y,\bar{y}] = \alpha y_t \bar{y}_t + \lambda y_{xx}\bar{y}_{xx} + \ell y_x \bar{y}_x + \ell^* y_x \bar{y}_x H(x-x_p) \tag{16}$$

The subscripts t and x indicate the partial derivatives of the functions y and $\bar{y}$.

Equation (16) can be integrated by parts. Two different forms of integration and end conditions are given. The first form of the integral is obtained by integrating by parts on the adjoint variable.

$$I = -\int_{t_o}^{t_b} \int_{x_o}^{x_b} \bar{y}Lydtdx + \int_{x_o}^{x_b} \alpha y_t y\Big|_{t_o}^{t_b} dx +$$

$$\int_{t_o}^{t_b} \{\lambda y_{xx}\bar{y}_x\Big|_{x_o}^{x_b} - (\lambda y_{xx})_x \bar{y}\Big|_{x_o}^{x_b} + \ell y_x y\Big|_{x_o}^{x_p} + \ell^* y_x y\Big|_{x_p}^{x_b}\}dt \tag{17}$$

where

$$Ly = (\alpha y_t)_t - (\lambda y_{xx})_{xx} + (\ell y_x)_x + (\ell^* y_x)_x H(x-x_p) \tag{18}$$

On the other hand, we can perform integration on the original variable to give

$$I = -\int_{t_o}^{t_b} \int_{x_o}^{x_b} \bar{\bar{y}}Lydtdx + \int_{x_o}^{x_b} \alpha y_t \bar{y}\Big|_{t_o}^{t_b} dx$$

$$+ \int_{t_o}^{t_b} \{\lambda \bar{y}_{xx}y_x\Big|_{x_o}^{x_b} - (\lambda \bar{y}_{xx})_x y\Big|_{x_o}^{x_b} + \ell \bar{y}_x y\Big|_{x_o}^{x_b} + \ell^* \bar{y}_x y\Big|_{x_p}^{x_b}\}dt \tag{19}$$

where

$$\bar{\bar{L}}y = (\alpha \bar{y}_t)_t - (\lambda \bar{y}_{xx})_{xx} + (\ell \bar{y}_x)_x + (\ell_p \bar{y}_x)_x H(x-x_a) \qquad (20)$$

For a fourth order spatial partial and a second order temporal partial system Eq. (10) becomes

$$D = \int_{x_o}^{x_b} \int_{t_o}^{t_b} y\bar{L}ydtdx - \int_{x_o}^{x_b} \int_{t_o}^{t_b} y\bar{\bar{L}}ydtdx \qquad (21)$$

By equating Eqs. (17) and (19) and solving for D in Eq. (21) we are converting the double integral into two single integrals in terms of the boundary conditions.

We can express the quantity D as the sum of three parts on end conditions D_1, D_2, and D_3 as

$$D = D_1 + D_2 + D_3 \qquad (22)$$

The terms in D_1 involve the initial conditions of y and $\bar{y}$ as

$$D_1 = \int_{x_o}^{x_b} \{\alpha y_t \bar{y}\Big|_{t_o}^{t_b} - \alpha \bar{y}_t y\Big|_{t_o}^{t_b}\}dx$$

$$= \int_{x_o}^{x_b} \{\alpha_b(y_{tb}\bar{y}_b - \bar{y}_{tb}y_b) - \alpha_o(y_{to}\bar{y}_o - \bar{y}_{to}y_o)\}dx \qquad (23)$$

The terms in D_2 involve the boundary conditions from the second partials of y and $\bar{y}$ as

$$D_2 = \int_{t_o}^{t_b} \{\ell y_x \bar{y}\Big|_{x_o}^{x_b} - \ell \bar{y}_x y\Big|_{x_o}^{x_b} + \ell^* y_x \bar{y}\Big|_{x_p}^{x_b} - \ell^* \bar{y}_x y\Big|_{x_p}^{x_b}\}dt$$

$$= \int_{t_o}^{t_b} \{\ell_b y_{xb}\bar{y}_b - \ell_o y_{xo}\bar{y}_o - \ell_b \bar{y}_{xb} y_b + \ell_o \bar{y}_{xo} y_o$$

$$+ \ell_b^* y_{xb}\bar{y}_b - \ell_p y_{xp}\bar{y}_p - \ell_b \bar{y}_{xb} y_b + \ell_p \bar{y}_{xp} y_p\}dt \qquad (24)$$

The terms in D_3 involve the boundary conditions from the fourth partials of y and $\bar{y}$ as

$$D_3 = \int_{t_o}^{t_b} \{\lambda y_{xx} \bar{y}_x \Big|_{x_o}^{x_b} - (\lambda y_{xx})_x \bar{y} \Big|_{x_o}^{x_b} - \lambda \bar{y}_{xx} y_x \Big|_{x_o}^{x_b} + (\ell \bar{y}_{xx})_x y \Big|_{x_o}^{x_b}\} dt$$

$$= \int_{t_o}^{t_b} \{\lambda_b y_{xxb} \bar{y}_{xb} - \lambda_o y_{xxo} \bar{y}_{xo} - (\ell y_{xx})_{xb} \bar{y}_b + (\lambda y_{xx})_{xo} \bar{y}_o$$

$$- \ell_b \bar{y}_{xxb} y_{xb} + \lambda_o \bar{y}_{xxo} y_{xo} + (\ell \bar{y}_{xx})_{xb} y_b - (\lambda \bar{y}_{xx})_{xo} y_o\} dt \qquad (25)$$

In order that $D \equiv 0$ in Eq. (22) it is sufficient that

$$D_1 \equiv 0 \qquad (26a)$$

$$D_2 \equiv 0 \qquad (26b)$$

and

$$D_3 \equiv 0 \qquad (26c)$$

END CONDITIONS FOR THE ADJOINT SYSTEMS

In order to satisfy the requirements in Eq. (26) we separate them again in three different parts.

(a) Let us assume that the adjoint variables are

$$\bar{y}_b = k_1 y_o \quad , \quad \bar{y}_o = k_1 y_b \qquad (27)$$

$$\bar{y}_{tb} = -\alpha_b^{-1} \alpha_o k_1 y_{to} \quad , \quad \bar{y}_{to} = -\alpha_o^{-1} \alpha_b k_1 y_{tb} \qquad (28)$$

where k_1 is a constant. The above adjoint boundary conditions satisfy the requirement that $D_1 = 0$ in Eq. (23).

(b) Let us assume the following adjoint variables

$$\bar{y}_b = k_2 y_b \qquad \bar{y}_o = \frac{k_1^2}{k_2} y_o \qquad \bar{y}_p = k_3 y_p \qquad (29)$$

$$\bar{y}_{xb} = k_2 y_{xb} \qquad \bar{y}_{xo} = \frac{k_1^2}{k_2} y_{xo} \qquad y_{xp} = k_3 y_{xp} \qquad (30)$$

Where Eq. (29) is inconsistent with Eq. (27) and k_2 is another constant. Equations (29) and (30) imply that $D_2 = 0$ in Eq. (24).

(c) The following boundary conditions are assumed

$$\bar{y}_o = \frac{k_1^2}{k_2} y_o \quad , \quad \bar{y}_{xo} = y_{xo} \quad , \quad \bar{y}_{xxo} = y_{xxo} \quad , \quad \bar{y}_{xxxo} = \left(\frac{k_1^2}{k_2}\right) y_{xxxo} \tag{31}$$

$$\bar{y}_b = k_2 y_b \quad , \quad \bar{y}_{xb} = y_{xb} \quad , \quad \bar{y}_{xxb} = y_{xxb} \quad , \quad \bar{y}_{xxxb} = k_2\, y_{xxxb} \tag{32}$$

Equations (31) and (32) satisfy Eq. (25). Thus $D_3 = 0$.

By giving the appropriate values of the adjoint variables in terms of the original variables one may find that the requirement $D \equiv 0$ can be satisfied. This leads to the condition in Eq. (6) that

$$\delta J = 0$$

for all arbitrary variations δy and $\delta \bar{y}$.

FIRST VARIATION

Since the variations $\delta \bar{y}$ and δy are independent to each other, the part of δJ in Eq. (6) with variation $\delta \bar{y}$ can be expressed as

$$\delta J(\delta \bar{y}) = \int_{x_o}^{x_b} \int_{t_o}^{t_b} \delta \bar{y} L y dt + \int_{x_o}^{x_b} \int_{t_o}^{t_b} \delta \bar{y} Q dt dx = 0 \tag{33}$$

Where Ly is given in Eq. (18) and contains second and fourth partial differentiations in y. It is intended to include only low order partial differentiations in $\delta J(\delta \bar{y})$. This can be achieved by considering the variations of the bilinear expression I given by Eqs. (15) and (16) as,

$$\delta J(\delta \bar{y}) = \int_{t_o}^{t_b} \int_{x_o}^{x_b} [\alpha y_t \delta \bar{y}_t + \lambda y_{xx} \delta \bar{y}_{xx} + \ell y_x \delta \bar{y}_x] dt dx$$
$$+ \int_{t_o}^{t_b} \int_{x_o}^{x_b} \ell_p y_x \delta \bar{y}_x dt dx \tag{34}$$

A different form of the above variation can be obtained from Eq. (17) as

$$\delta I(\delta\bar{y}) = -\iint \delta\bar{y}Lydtdx + \int_{x_o}^{x_b} \alpha y_t \delta\bar{y}\Big|_{t_o}^{t_b} dx$$

$$+ \int_{t_o}^{t_b} \{\lambda y_{xx}\delta\bar{y}_x\Big|_{x_o}^{x_b} - (\lambda y_{xx})_x\delta\bar{y}\Big|_{x_o}^{x_b} + \ell y_x\delta\bar{y}\Big|_{x_o}^{x_b} + \ell^* y_x\delta\bar{y}\Big|_{x_p}^{x_b}\}dt \qquad (35)$$

Equating Eqs. (34) and (35), solving for the term containing integrals for $\delta\bar{y}Ly$ and substituting into Eq. (33) we have

$$\delta J(\delta\bar{y}) = \int_{x_o}^{x_b} (\alpha y_t)\delta\bar{y}\Big|_{t_o}^{t_b} dx + \int_{t_o}^{t_b} \lambda y_{xx}\delta\bar{y}_x\Big|_{x_o}^{x_b} dt$$

$$+ \int_{t_o}^{t_b} \{[\ell y_x - (\lambda y_{xx})_x]\delta\bar{y}\Big|_{x_o}^{x_b} + \ell^* y_x\delta y\Big|_{x_p}^{x_b}\}dt + \int_{t_o}^{t_b}\int_{x_o}^{x_b} \delta\bar{y}Qdtdx$$

$$- \int_{t_o}^{t_b}\int_{x_o}^{x_b} \{\alpha y_t\delta\bar{y}_t + \lambda y_{xx}\delta\bar{y}_{xx} + \ell y_x\delta\bar{y}_x + \ell^* y_x\delta\bar{y}_x H(x-x_p)\}dtdx \doteq 0 \quad (36)$$

This is the key equation which uses variational principle in solving a mixed initial and boundary value problem for a fourth order partial differential equation.

DISCUSSION OF THE VARIATIONAL EQUATION

Let us discuss the various terms in Eq. (36), the variational equation for the beam problem, into three parts as follows.

(1) The initial conditions of the original variables are given and variations of the adjoints at the far end are zero. The first term in Eq. (36) contains the product of $y_t\delta\bar{y}$ evaluated at the initial condition $y_{to}\delta\bar{y}_o$ and at the final condition $y_{tb}\delta\bar{y}_b$. Since the value of y_b are known as given by Eqs. (27) and (29), $\delta\bar{y}_b = 0$. That is, the variations of the adjoint variable at the far end are zero.

(2) The boundary conditions of the original variables and variation of the adjoints can be determined. The second through fourth terms are the boundary terms involving the variations $\delta\bar{y}$ and $\delta\bar{y}_x$ and the variables y_x, y_{xx}, and y_{xxx} at both boundaries. For a beam the end conditions can be expressed as

Fixed End	$y = 0$	$\bar{y} = 0$	$\delta\bar{y} = 0$
	$y_x = 0$	$\bar{y}_x = 0$	$\delta\bar{y}_x = 0$
Hinged End	$y = 0$	$\bar{y} = 0$	$\delta\bar{y} = 0$
	$y_{xx} = 0$	$\bar{y}_{xx} = 0$	$\delta\bar{y}_{xx} = 0$
Guided End	$y_x = 0$	$\bar{y}_x = 0$	$\delta\bar{y}_x = 0$
	$y_{xxx} = 0$	$\bar{y}_{xxx} = 0$	$\delta\bar{y}_{xxx} = 0$
Free End	$y_{xx} = 0$	$\bar{y}_{xx} = 0$	$\delta\bar{y}_{xx} = 0$
	$y_{xxx} = 0$	$\bar{y}_{xxx} = 0$	$\delta\bar{y}_{xxx} = 0$

The variations in the adjoint variables shown in the last column coincide to the same end conditions in the original variables given in the first column, whether it is on the left or the right boundary. It is noted that the third partial derivatives can be evaluated at the boundaries.

(3) Interior region - The last two terms give the interior where the forcing function Q, the adjoint-variations $\delta\bar{y}$, $\delta\bar{y}_t$, $\delta\bar{y}_x$, and $\delta\bar{y}_{xx}$ and the variables y_t, y_x, and y_{xx} are shown. No third order partial of y with respect to x is present. Thus the variables that are needed for the computation are y, y_t, y_x, y_{xt}, y_{xx}, and y_{xxt}. This requires a c^2 continuity in the spatial direction and a c^1 continuity in the time domain.

TRANSFORMATION OF COORDINATES

The integral signs in Eq. (36) can be converted into summation signs if discrete intervals for integration are used. We may take some scale factors to nondimensionalize the problem by giving

$$t_o = 0 \quad , \quad t_b = 1 \qquad 0 < t < 1 \tag{37}$$

$$x_o = 0 \quad , \quad x_b = 1 \qquad 0 < x < 1 \tag{38}$$

Moreover, Eq. (36) can be discretized by letting

$$\xi = Ht - i+1 \qquad 0 < \xi < 1 \qquad i = 1,2,\ldots,H \tag{39}$$

$$\eta = Kx - j+1 \qquad 0 < \eta < 1 \qquad j = 1,2,\ldots,K \tag{40}$$

where H and K are number of intervals for t and x respectively. Thus the partial derivatives are

$$y_t = \frac{\partial y}{\partial t} = H \frac{\partial y}{\partial \xi} = Hy_\xi \tag{41}$$

$$y_x = \frac{\partial y}{\partial x} = K \frac{\partial y}{\partial \eta} = Ky_\eta \tag{42}$$

$$y_{xx} = \frac{\partial^2 y}{\partial x^2} = K \frac{\partial y_x}{\partial \eta} = K^2 y_{\eta\eta} \tag{43}$$

$$y_{xxx} = \frac{\partial^3 y}{\partial x^3} = K \frac{\partial y_{xx}}{\partial \eta} = K^3 y_{\eta\eta\eta} \tag{44}$$

Use of Eqs. (36) through Eq. (44) then leads to

$$
\begin{aligned}
0 &= \delta J(\delta \bar{y}) \\
&= \sum_{j=1}^{K} \int_0^1 [\alpha H y_\xi(i,j)]\delta\bar{y}(i,j)\Big|_{t_o}^{t_b} \frac{1}{K}\, d\eta \\
&+ \sum_{i=1}^{H} \int_0^1 [\ell K y_\eta - (\lambda K^3 y_{\eta\eta})_\eta]\delta\bar{y}(i,j)\Big|_{x_o}^{x_b} \frac{1}{H}\, d\xi
\end{aligned}
$$

$$+ \sum_{i=1}^{H} \int_0^1 (\lambda K^2 y_{\eta\eta}) K\delta\bar{y}_\eta(i,j) \Big|_{x_o}^{x_b} \frac{1}{H}\, d\xi$$

$$+ \sum_{i=p}^{H} \int_0^1 \ell^* K y_\eta \delta\bar{y}(i,j) \Big|_{x_p}^{x_b} \frac{1}{H}\, d\xi$$

$$+ \sum_{j=1}^{K} \int_0^1 \{ \sum_{i=1}^{H} \int_0^1 \delta\bar{y}(i,j) Q \frac{1}{H}\, d\xi \} \frac{1}{K}\, d\eta$$

$$- \sum_{j=1}^{K} \int_0^1 \{ \sum_{i=1}^{H} \int_0^1 [\alpha H^2 y_\xi(i,j) \delta\bar{y}_\xi(i,j) + \lambda K^4 y_{\eta\eta} \delta\bar{y}_{\eta\eta} + \ell K^2 y_\eta \delta\bar{y}_\eta] \frac{1}{H}\, d\xi \} \frac{1}{K}\, d\eta$$

$$- \sum_{i=p}^{K} \int_0^1 \{ \sum_{i=1}^{H} \int_0^1 [\ell^* K^2 y_\eta \delta\bar{y}_\eta H(x-x_p)] \frac{1}{H}\, d\xi \} \frac{1}{K}\, d\eta = 0 \qquad (45)$$

GRID SYSTEMS

The (24x1) vector $Y^{(i,j)}$ has a grid of four (6x1) vectors $Y_1^{(i,j)}$ through $Y_4^{(i,j)}$, thus

$$Y^{(i,j)} = \{[Y_1^{(i,j)}]^T [Y_2^{(i,j)}]^T [Y_3^{(i,j)}]^T [Y_4^{(i,j)}]^T \qquad (46)$$

Each of the (6x1) vectors has six components consisting of the function, its first and second partials in spatial directions, and its mixed partials in space and time.

$$Y_1^{(i,j)} = \begin{bmatrix} y(\xi_i,\eta_j) \\ y_\xi(\xi_i,\eta_j) \\ y_\eta(\xi_i,\eta_j) \\ y_{\xi\eta}(\xi_i,\eta_j) \\ y_{\eta\eta}(\xi_i,\eta_j) \\ y_{\xi\eta\eta}(\xi_i,\eta_j) \end{bmatrix} \qquad Y_3^{(i,j)} = \begin{bmatrix} y(\xi_i,\eta_{j+1}) \\ y_\xi(\xi_i,\eta_{j+1}) \\ y_\eta(\xi_i,\eta_{j+1}) \\ y_{\xi\eta}(\xi_i,\eta_{j+1}) \\ y_{\eta\eta}(\xi_i,\eta_{j+1}) \\ y_{\xi\eta\eta}(\xi_i,\eta_{j+1}) \end{bmatrix}$$

$$
Y_2(i,j) = \begin{bmatrix} y(\xi_{i+1},\eta_j) \\ y_\xi(\xi_{i+1},\eta_j) \\ y_\eta(\xi_{i+1},\eta_j) \\ y_{\xi\eta}(\xi_{i+1},\eta_j) \\ y_{\eta\eta}(\xi_{i+1},\eta_j) \\ y_{\xi\eta\eta}(\xi_{i+1},\eta_j) \end{bmatrix} \qquad
Y_4(i,j) = \begin{bmatrix} y(\xi_{i+1},\eta_{j+1}) \\ y_\xi(\xi_{i+1},\eta_{j+1}) \\ y_\eta(\xi_{i+1},\eta_{j+1}) \\ y_{\xi\eta}(\xi_{i+1},\eta_{j+1}) \\ y_{\eta\eta}(\xi_{i+1},\eta_{j+1}) \\ y_{\xi\eta\eta}(\xi_{i+1},\eta_{j+1}) \end{bmatrix} \tag{47}
$$

If we increase the row index from i to i+1, then the grid point shifts down by one step and the following holds

$$
Y_1(i+1,j) = Y_2(i,j) \qquad Y_3(i+1,j) = Y_4(i,j) \tag{48}
$$

If we increase the column index from j to j+1 then the grid point shifts to the right by one step and one obtains

$$
Y_1(i,j+1) = Y_3(i,j) \qquad Y_2(i,j+1) = Y_4(i,j) \tag{49}
$$

The following diagram shows the relationship of the grid system.

$$
\begin{array}{ccccccc}
 & & & \vdots & & & \\
 & Y_1(i,j) & & Y_3(i,j) = Y_1(i,j+1) & & Y_3(i,j+1) & \\
 & & [Y(i,j)] & & [Y(i,j+1)] & & \\
 & Y_2(i,j) & & Y_4(i,j) = Y_2(i,j+1) & & Y_4(i,j+1) & \\
\cdots & \| & & \| \qquad\quad \| & & \| & \cdots \\
 & Y_1(i+1,j) & & Y_3(i+1,j) = Y_1(i+1,j+1) & & Y_3(i+1,j+1) & \\
 & & [Y(i+1,j)] & & [Y(i+1,j+1)] & & \\
 & Y_2(i+1,j) & & Y_4(i+1,j) = Y_2(i+1,j+1) & & Y_4(i+1,j+1) & \\
 & & & \vdots & & &
\end{array}
$$

SPLINE FUNCTION

We may express the variables $y^{(i,j)}$ and $\delta\bar{y}^{(i,j)}$ in Eq. (45) in terms of the (1x24) spline function $a^T(\xi,\eta)$ and the (24x1) node point function $Y^{(i,j)}$ as follows.

$$y^{(i,j)}(\xi,\eta) = a^T(\xi,\eta)Y^{(i,j)} \tag{50}$$

where

$$a^T(\xi,\eta) = \{[a^1(\xi,\eta)]^T \; [a^2(\xi,\eta)]^T \; [a^3(\xi,\eta)]^T \; [a^4(\xi,\eta)]^T \tag{51}$$

and

$$\delta\bar{y}^{(i,j)}(\xi,\eta) = a^T(\xi,\eta)\delta\bar{Y}^{(i,j)} \tag{52}$$

A typical term for a product can be written as

$$\delta\bar{y}^{(i,j)}y^{(i,j)} = [\delta\bar{Y}^{(i,j)}]^T a(\xi,\eta)a^T(\xi,\eta)Y^{(i,j)} \tag{53}$$

CONCLUSION

A bilinear form of the original and adjoint variable is employed in determining the coefficients of the variations of the functions and their first derivatives. There is no term involving the variations of any higher derivatives than second. In solving mixed boundary and initial value problems of a fourth order partial differential equation using spline functions, the computation may be simplified considerably if the variable in time can be truncated into arbitrary sections. The entire problem is divided into several strips of distinct time intervals, each strip containing mostly the boundary value problem.

The variational principle for spatial and temporal problems with boundary and initial conditions have been investigated. This variational principle is very general in scope and can be applied to many linear partial differential

equations. The principle is applicable if the bilinear concomitant is identically zero. This leads to the requirement that a set of end conditions for the adjoint systems must be found to satisfy this condition. Otherwise the variational principle as stated may not be applicable.

The beam equation (with one dimensional spatial direction) satisfy these variational principles. For future work the analytic solution of these equations using finite element method will be studied. The assembly of the elements of the matrices involved in the formulation will be demonstrated. The stability problem in numerical solutions on these equations will also be investigated. This lays the foundation for the gun dynamics problem to be studied in the future.

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APPENDIX

THE VARIATIONAL PRINCIPLE

A more accurate estimate can be made by constructing a variational principle.[5] By using the adjoint variable y as a Lagrange multiply we have

$$J[y,\bar{y}] = \langle \bar{Q}y \rangle + \langle \bar{y},(Q+Ly) \rangle$$

$$= \langle \bar{Q},y \rangle + \langle \bar{y},Q \rangle + \langle \bar{y},Ly \rangle \qquad \text{(A1)}$$

In order that J be a variational principle the following requirements must be satisfied.

(a) J is stationary about the function y_s which satisfies the following relation

$$Ly_s = -Q \qquad \text{(A2)}$$

(b) The stationary value of J deduced from Eqs. (2) through (5) is

$$J[y,\bar{y}] = \langle \bar{Q},y_s \rangle + \langle \bar{Q},y_a \rangle \qquad \text{(A3)}$$

where y_a is the actual solution. Consider first the stationarity of J by taking the variation of Eq. (A1)

$$\delta J = \langle \bar{Q},\delta y \rangle + \langle \delta\bar{y},Q \rangle + \langle \delta\bar{y},Ly \rangle + \langle \bar{y},L\delta y \rangle$$

$$= \langle \delta\bar{y},(Ly+Q) \rangle + \langle \delta y,(\overline{Ly}+\bar{Q}) \rangle$$

$$- \langle \delta y,\overline{Ly} \rangle + \langle \bar{y},L\delta y \rangle \qquad \text{(A4)}$$

We will make an effort later to impose certain conditions in order that the following equality holds:

$$\langle \bar{y},L\delta y \rangle = \langle \delta y,\overline{Ly} \rangle \qquad \text{(A5)}$$

where $\bar{L}$ is the adjoint operator.

---

[5] Stacey, W. M. Jr., Variational Methods in Nuclear Reactor Physics, Academic Press, 1974.

By combining Eqs. (A4) and (A5) one obtains

$$\delta J = \langle \delta\bar{y},(Ly+Q)\rangle + \langle \delta y,(\bar{L}\bar{y}+\bar{Q})\rangle = 0 \tag{A6}$$

Since the variations $\delta\bar{y}$ and δy are arbitrary it leads to the requirement that the stationary values y_s and ys must satisfy

$$Ly_s = -Q \tag{A7}$$

$$\bar{L}\bar{y}_s = -\bar{Q} \tag{A8}$$

Since Eq. (A7) is the same as Eq. (A2) therefore, J is stationary about the function y_s.

Equation (A8) is the adjoint equation in terms of the adjoint operator, $\bar{L}$, the adjoint variable $\bar{y}$, and the adjoint forcing function $\bar{Q}$.

It is noted that δJ in Eq. (A6) vanishes and is independent of the arbitrary variations δy and $\delta\bar{y}$. By using δJ one can claim that the estimate is very accurate and free from the arbitrary variations.

Using the relationship in Eq. (A7) the stationary value of J from Eq. (A1) is

$$J[\bar{y}_s,y_s] = \langle\bar{Q},y_s\rangle + \langle\bar{y}_s,Q\rangle + \langle\bar{y}_s,Ly_s\rangle = \langle\bar{Q},y_s\rangle \tag{A9}$$

Since J is stationary and $\delta J \to 0$, then

$$\langle\bar{Q},y_s\rangle \to \langle\bar{Q},y_a\rangle \tag{A10}$$

which is the requirement given in Eq. (A3).

It is noted that Eq. (A6) contains no boundary terms to be satisfied. This bears an important point in the future discussion.

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